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The Identity Theorem for Analytic Functions

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It is well known [1, p. 24] that a kth degree polynomial

$$p_k(z) = a_{k,0} + \cdots + a_{k,n}z^n + \cdots + a_{k,k}z^k \tag{1}$$

over a field (say, the field of complex numbers) is uniquely determined by $k+1$ distinct elements $c_0, \ldots, c_k$ of the field and the corresponding $k+1$ values $p_k(a_0), \ldots, p_k(c_k)$ of the polynomial $p_k(z)$. In fact, for the coefficient $a_{k,n}$ in (1) we have:

$$a_{k,n} = \frac{\begin{vmatrix} 1 & c_0 & c_0^2 & \cdots & p_k(c_0) & \cdots & c_0^k \\ 1 & c_1 & c_1^2 & \cdots & p_k(c_1) & \cdots & c_1^k \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & c_k & c_k^2 & \cdots & p_k(c_k) & \cdots & c_k^k \end{vmatrix}}{\begin{vmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n & \cdots & c_0^k \\ 1 & c_1 & c_1^2 & \cdots & c_1^n & \cdots & c_1^k \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & c_k & c_k^2 & \cdots & c_k^n & \cdots & c_k^k \end{vmatrix}} = \frac{\det(1, c_i, c_i^2, \ldots, p_k(c_i), \ldots, c_i^k)}{\det(1, c_i, c_i^2, \ldots, c_i^n, \ldots, c_i^k)}. \tag{2}$$

In this paper we generalize the above result to the case of analytic functions. Specifically, given a sequence of complex numbers $c_0, c_1, c_2, \ldots$ converging to 0, and the corresponding values $f(c_0), f(c_1), f(c_2), \ldots$ of an analytic function f we determine the nth coefficient a_n of the Taylor expansion $\sum_{n=0}^{\infty} a_n z^n$ of f as the limit of the quotients of appropriate determinants. In fact,

$$a_n = \lim_{k\to\infty} \frac{A_k}{B_k},$$

where A_k and B_k are respectively the numerator and the denominator of (2) corresponding to the kth interpolation polynomial $p_k(z)$ of f. Thus, $p_k(z)$ is the unique polynomial of kth degree which takes on the values $f(c_0), f(c_1), \ldots, f(c_k)$ respectively for the complex numbers $c_0, c_1, \ldots, c_k$.

Denoting by $\det(p_i, q_i, \ldots)$ the determinant whose ith row is $(p_i, q_i, \ldots)$, we prove the following theorem.

THEOREM. *Let $(c_i)_{i=0,1,2,\ldots}$ be a sequence of pairwise distinct complex numbers c_i converging to 0. Let f be an analytic function in a disk whose center is at 0 and which contains c_i's. Then in the disk*

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \tag{3}$$

where

$$a_n = \lim_{k\to\infty} \frac{\det(1, c_i, c_i^2, c_i^3, \ldots, f(c_i), \ldots, c_i^k)}{\det(1, c_i, c_i^2, c_i^3, \ldots, c_i^n, \ldots, c_i^k)}. \tag{4}$$

Proof. We observe that since c_i's are pairwise distinct, the $k+1$ by $k+1$ determinant appearing in the denominator of (4) is never zero.

Let $p_k(z)$ be the kth degree interpolation polynomial of f. Thus, by [1, p. 79], we have

$$p_k(c_i) = f(c_i) \qquad \textit{for } i = 0, \ldots, k. \tag{5}$$

From Theorem 4.3.1 of [1, p. 81] it follows that in the disk, under the hypothesis of the theorem, the interpolation polynomials $p_k(z)$ converge uniformly to $f(z)$. But then the sequence of the nth derivative polynomials $p_k^{(n)}(z)$ also converge to the nth derivative function $f^{(n)}(z)$. Consequently,

$$\lim_{k\to\infty} p_k^{(n)}(z) = f^{(n)}(z)$$

from which it follows

$$\lim_{k\to\infty} p_k^{(n)}(0) = f^{(n)}(0). \tag{6}$$

Letting $0! = 1$ and identifying a function with its 0th derivative, from (6), (1), and (3) we obtain

$$\lim_{k\to\infty} n!\, a_{k,n} = n!\, a_n,$$

and, therefore,

$$\lim_{k\to\infty} a_{k,n} = a_n. \tag{7}$$

But then (5), (2), and (7) imply (4), as desired.

REFERENCE

1. P. J. DAVIS, "Interpolation and Approximation," Blaisdell Publishing Co., New York, 1963.